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EXCITED STATE DESTRI – DE VEGA EQUATION FOR SINE–GORDON AND RESTRICTED SINE–GORDON MODELS

D. Fioravanti, A. Mariottini, E. Quattrini and F. Ravanini

*Sez. I.N.F.N. and Dip. di Fisica - Univ. di Bologna
Via Irnerio 46, I-40126 BOLOGNA, Italy*

Abstract

We derive a generalization of the Destri - De Vega equation governing the scaling functions of some excited states in the Sine-Gordon theory. In particular configurations with an even number of holes and no strings are analyzed and their UV limits found to match some of the conformal dimensions of the corresponding compactified massless free boson. Quantum group reduction allows to interpret some of our results as scaling functions of excited states of Restricted Sine-Gordon theory, i.e. minimal models perturbed by ϕ_{13} in their massive regime. In particular we are able to reconstruct the scaling functions of the off-critical deformations of all the scalar primary states on the diagonal of the Kac-table.

1 Introduction

The exact computation of scaling functions in two-dimensional integrable quantum field theories has been paid a lot of attention in the last years. The Thermodynamic Bethe Ansatz (TBA) methods [1, 2], for example, are able to provide, for a theory whose factorizable S-matrix is known, a very good framework to compute a set of coupled non-linear integral equations driving the evolution along the Renormalization Group flow of the Casimir energy of the vacuum on a cylinder. Towards the UV limit this quantity is directly related to the central charge of the corresponding Conformal Field Theory.

However, the TBA method suffers of some disadvantages. First of all the deduction of the integral equations is done starting from the Factorized Scattering Theory (FST) and not from the Quantum Field Theory (QFT) itself. Now, the FST is often only *conjectured* and the TBA itself is therefore not fully *deduced* from the QFT. Another point is that there are models where the number of equations for the TBA becomes infinite or increases to a point that numerical integration becomes untractable.

An undoubtable progress of these exact methods came with the introduction, at least for the Sine-Gordon model, of the so called Destri – De Vega (DdV) equation [3, 4], whose major advantages are:

- to summarize in one complex integral equation a large number, sometimes an infinity, of real coupled integral equations of TBA type;
- to be deduced from a lattice regularization of the model itself and not from the scattering theory corresponding to it.

Unfortunately the DdV equation is developed and well explored only for the vacuum of the Sine-Gordon model and it would be very important to write down analogous equations for other integrable field theories too, as well as for excited states. In this letter we make progress towards the study of DdV equation for excited states in the Sine-Gordon model. The quantum group reduction allows then to extend these results to the Restricted Sine-Gordon models, i.e. the integrable theories obtained by perturbing the conformal minimal models by their relevant operator ϕ_{13} . This is the first case of extension of the DdV equation out of the pure Sine-Gordon realm.

In sect.2, after recalling some general properties of the SG model on a cylinder, we write down, following [4, 5], the DdV equation for vacua with twisted boundary conditions. These vacua are states that can be obtained from the true vacuum by application of *spin fields*. In sect.3 we observe that after quantum group reduction, some of these states can be reinterpreted in the framework of RSG theory as the off-critical deformations of scalar primary fields along the main diagonal of the Kac-table. A very precise numerical check against known cases confirms this procedure. In sect.4 we afford the problem of writing the DdV equation for excited states over the twisted vacua, by exciting Bethe strings and creating holes over the Dirac sea of real solutions. This corresponds to approaching the problem in a

true Bethe ansatz spirit. In this short letter we shall make the choice to illustrate for technical simplicity only the case of pure *hole* excitations, leaving the more general problem of string excitations to a future, more extensive publication [6]. In sect.5 we compute the energy as a functional of the solution of the DdV equation. In sect.6 we investigate the ultraviolet (UV) limit of the energy and compare with the operator content of the compactified free boson CFT that represents the UV limit of SG theory. Some preliminary numerical investigations are reported in sect.7 and in sect.8 we draw our conclusions and perspectives for future work.

2 The Sine-Gordon Hilbert Space on a cylinder

The Sine-Gordon (SG) model, with action

$$S = \int d^2x \left\{ \frac{1}{2} : (\partial\phi)^2 : + \lambda : \cos \beta\phi : \right\} \quad (1)$$

is invariant under the symmetry $\phi \rightarrow \phi + 2\pi n/\beta$. States transforming as $|\Psi\rangle \rightarrow e^{i\alpha}|\Psi\rangle$ under this symmetry constitute a so-called α -sector \mathcal{H}_α of the Hilbert Space of states $\mathcal{H} = \bigoplus_{-\pi < \alpha \leq \pi} \mathcal{H}_\alpha$. On a cylinder of radius R the degeneracy among the α -vacua is removed and the various α -sectors are selected by imposing suitable twisted boundary conditions on the space direction of the cylinder. For $p = \frac{\beta^2}{8\pi - \beta^2}$ rational, the values of α are quantized; if p is integer, i.e. the case that we shall mainly consider in the following, then $\alpha = \frac{\pi\kappa}{p}$, $\kappa = 0, 1, \dots, p$. The α -sectors have of course a state of lowest energy, the so called α -vacuum. For $\alpha = 0$ this is the true vacuum of the theory, the other α -vacua can be excited from the true vacuum by particular operators which are called *spin operators*. In the CFT representing the ultraviolet (UV) limit of SG at a fixed p (SG_p) (a $c = 1$ massless free boson compactified on a circle of radius $\frac{\sqrt{4\pi}}{\beta}$) these operators are represented by vertex operators of conformal dimensions $\Delta_\kappa = \frac{\kappa^2}{4p(p+1)}$. For more details see, e.g., ref. [7].

A DdV equation can easily be written for all the α -vacua along the lines of the standard deduction of the original work of Destri and De Vega [3, 4]. Indeed, one simply has to consider twisted boundary conditions, for which a DdV equation is written in [4] for the XXZ chain. It is a straightforward exercise to rewrite it in the form more appropriate for the SG_p model. The resulting equation is [5]

$$f_\alpha(\theta) = ir \sinh \theta + i\alpha + 2i \int_{-\infty}^{+\infty} d\theta' G(\theta - \theta') \Im m \log(1 + e^{f_\alpha(\theta + i0)}) \quad (2)$$

where $r = mR$, m being the soliton mass. The kernel $G(\theta) = -\frac{i}{2\pi} \frac{d}{d\theta} \log S(\theta)$ is determined by the soliton-soliton scattering amplitude

$$S(\theta) = \exp \frac{i}{2} \int_{-\infty}^{+\infty} dk \frac{\sin(\theta k)}{k} \frac{\sinh \frac{k\pi}{2}(p-1)}{\sinh \frac{k\pi}{2} \cosh \frac{k\pi}{2}} \quad (3)$$

Once the pseudoenergy $f_\alpha(\theta)$ is determined by solving the DdV equation, it can be used to compute the ground state Casimir energy $E_\alpha(R) = -\frac{\pi c_\alpha(r)}{6R}$, where

$$c_\alpha(r) = \frac{3r}{\pi^2} \int_{-\infty}^{+\infty} d\theta \sinh \theta \Im m \log(1 + e^{f_\alpha(\theta + i0)}) \quad (4)$$

The adimensional function $c_\alpha(r)$ is known as scaling function for the α -sector ground state. A standard dilogarithm calculation shows that the UV limit $r \rightarrow 0$ of this function is given by

$$c_\alpha(0) = c_{UV} - 12(\Delta_+ - \Delta_-) = 1 - \frac{6p}{p+1} \left(\frac{\alpha}{\pi}\right)^2 \quad (5)$$

thus reproducing the correct central charge $c = 1$ of the free boson and the correct conformal dimensions Δ_+ and Δ_- of the spin operators mentioned before.

One can thus follow the renormalization group destiny of the α -vacua in the different sectors on the cylinder in an exact way, overcoming the need of approximante methods as the Truncated Conformal Space Approach (TCSA) [8]. The obvious result that one can see from a plot of the functions $c_\alpha(r)$ for all α 's is that they all accumulate towards infrared to the vacuum of the theory.

3 RSG_p spin states

An interesting application of this result is the possibility to reinterpret it in the light of the quantum group truncation occurring at rational p , which, as shown by [9, 10], selects inside the Hilbert space of the full SG_p model, a certain subspace with a consistent subalgebra of operators acting on it in such a way that it can be thought as a sort of smaller theory embedded in the larger SG_p one. The SG vacuum never appears as a state in this *restricted* Sine-Gordon (RSG_p) model. It is known that the role of the vacuum is instead played in this context by the α -vacuum with $\kappa = 1$ (we restrict here for simplicity to the integer p case, it is not difficult to generalize to any rational p). Indeed, $\lim_{r \rightarrow 0} c_{\pi/p}(r) = 1 - \frac{6}{p(p+1)}$, as it must be for the correct scaling function of the vacuum of the corresponding minimal model perturbed by $\phi_{1,3}$. To support this result we made numerical comparison of the scaling function obtained by this DdV equation with that obtained from the traditional TBA of ref. [11]. The two functions agree with 14 significant digits, for integer values of p ranging from 3 to 12.

The quantum group truncation eliminates only two α -vacua: the $\kappa = 0, \pi$ ones. All the other α -vacua have an interpretation as RSG_p states, namely they correspond to the (off-critical evolution of) the primary states on the diagonal of the Kac-table of the p -th minimal model, the so called *spin states*. Indeed the computation of the UV limit of the scaling function gives in general $c_{\kappa\pi/p}(0) = 1 - \frac{6}{p(p+1)} - 24\frac{\kappa^2-1}{4p(p+1)}$, thus providing the conformal dimensions of the states in the diagonal of the Kac-table. The scaling functions of a few of these states were already explored by other authors with other means [12, 13, 14]. We succeeded in various cases to compare our numerical results with theirs, getting the same kind of good agreement up to 14 significant digits as for the vacuum.

The quantum group reduction for other states, especially for secondaries, is more involved and a full treatment goes out of the scope of the present letter. Let us only comment here that unfortunately none of the hole excitations we are going to describe in the following is in

the class (type II representations) that survives quantum group reduction.

4 DdV for pure hole excited states

In this section we present our derivation of the DdV equation describing *pure hole* excitations obtained from the vacuum by creating some holes in the distribution of real roots.

Let us consider the Bethe equations for the SG_p model discretized as a 6-vertex model on a light-cone lattice with $2N$ sites. Also, take the corresponding inhomogeneous spin chain have ω -twisted boundary conditions, related to the previous α by $\alpha = -\omega \frac{p+1}{2p}$. These Bethe equations have the form

$$\prod_{m=1}^M \frac{\sinh(\lambda_j - \lambda_m + i\gamma)}{\sinh(\lambda_j - \lambda_m - i\gamma)} = - \prod_{n=1}^{2N} \frac{\sinh(\lambda_j + i\Theta_n + \frac{i\gamma}{2})}{\sinh(\lambda_j + i\Theta_n - \frac{i\gamma}{2})} e^{i\omega} \quad (6)$$

Here $\gamma = \frac{\pi}{p+1}$. The fundamental state (α -vacuum) is the one with $M = N$. The light-cone approach corresponds to the choice of inhomogeneities $i\Theta_n = (-1)^{n+1}\Theta$. Define the function

$$\phi(\lambda, x) = i \log \frac{\sinh(ix + \lambda)}{\sinh(ix - \lambda)} \quad (7)$$

and the M real solutions counting function

$$Z_M(\lambda) = N[\phi(\lambda + \Theta, \frac{\gamma}{2}) + \phi(\lambda - \Theta, \frac{\gamma}{2})] - \sum_{k=1}^M \phi(\lambda - \lambda_k, \gamma) - \omega \quad (8)$$

Then, the logarithm of the Bethe equations can be written as¹

$$Z_M(\lambda_j) = 2\pi I_j \quad , \quad I_j \in \mathbf{Z} + \frac{1}{2} \quad , \quad j = 1, \dots, M \quad (9)$$

We consider here a configuration with $M < N$ real solutions and H holes (i.e. H values of I_j are missing). Consistency of the Bethe equations requires that the number H must be even [15, 16]. The range of possible values of I_j actually increases due to this ‘‘perturbation’’ of the system. Now there are in total $T = M + H = N + \frac{H}{2}$ possibilities, while the roots are $M = N - \frac{H}{2}$. In order to implement the usual DdV trick, we have to rewrite the counting function $Z_M(\lambda)$ by adding and subtracting the contribution of holes

$$Z_M(\lambda) = NQ(\lambda) - \omega - \sum_{k=1}^T \phi(\lambda - \lambda_k, \gamma) + \sum_{h=1}^H \phi(\lambda - \lambda_h, \gamma) \quad (10)$$

where we have indicated for short $Q(\lambda) = \phi(\lambda + \Theta, \frac{\gamma}{2}) + \phi(\lambda - \Theta, \frac{\gamma}{2})$. The standard derivation of DdV equations makes use of the trick

$$\phi(\lambda - \lambda_j, \gamma) = \oint_{\lambda_j} \frac{d\mu}{2\pi i} \phi(\lambda - \mu) \frac{d}{d\mu} \log(1 + e^{iZ_M(\mu)}) \quad (11)$$

¹In this paper we are interested only in states with M even

to write (the integration path Γ contours the whole real axis counterclockwise)

$$\sum_{k=1}^T \phi(\lambda - \lambda_k, \gamma) = \oint_{\Gamma} \frac{d\mu}{2\pi i} \phi(\lambda - \mu) \frac{d}{d\mu} \log(1 + e^{iZ_M(\mu)}) \quad (12)$$

$$= i(X * L)(\lambda) + i(X * Z_M)(\lambda) \quad (13)$$

where $L(\lambda) = \log(1 + e^{iZ_M(\lambda+i\eta)}) - \log(1 + e^{-iZ_M(\lambda-i\eta)})$ for $\eta \rightarrow 0$, $*$ denotes standard convolution $(f * g)(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)dy$ and we have introduced the notation $X(\lambda) = \frac{1}{2\pi} \phi'(\lambda, \gamma)$.

The rest of the derivation repeats the same steps as the standard one for the vacuum DdV [3, 4], with the only modification that $NQ(\lambda)$ must be now substituted with $A \equiv NQ - \omega + \sum_{h=1}^H \phi(\lambda - \lambda_h, \gamma)$. This goes to modify the $imR \sinh \theta$ term because instead of computing $N(K * Q)(\lambda)$ we have to compute now $(K * A)(\lambda)$, where $K(\lambda)$ is the inverse of $(\delta + X)(\lambda)$, that is

$$[K * (\delta + X)](\lambda) = \delta(\lambda) \quad (14)$$

We have to compute terms like $K * \phi(\lambda - \mu, \gamma)$. Just doing a Fourier transform we get in the case of μ real:

$$K * \phi(\lambda - \mu, \gamma) = -i \log S(\frac{\pi}{\gamma}(\lambda - \mu)) \quad (15)$$

All these calculations are easily performed in Fourier transformed space, by remembering that the Fourier Transform of $\phi(\lambda, \gamma)$ is given by $\tilde{\phi}(k, \gamma) = \text{sign}(\gamma) \frac{2\pi \sinh(k(\frac{\pi}{2} - |\gamma|))}{ik \sinh \frac{k\pi}{2}}$.

The continuum form of DdV equation is recovered if we let $\theta = \lambda\pi/\gamma$, $\alpha = -\omega \frac{p+1}{2p}$ and $f(\theta) = \lim_{N \rightarrow \infty} iZ_M(\lambda)$

$$\begin{aligned} f(\theta) &= imR \sinh \theta + i\alpha + \sum_{h=1}^H \log S(\theta - \theta_h) \\ &+ 2i \int_{-\infty}^{+\infty} d\theta' G(\theta - \theta') \Im m \log(1 + e^{f(\theta' + i\eta)}) \end{aligned} \quad (16)$$

where $\theta_h = \frac{\pi}{\gamma} \lambda_h$

The condition we use to determine the parameters θ_h is the fact that they are *missing roots*, i.e. although they are not taken as roots of the Bethe equations, nevertheless they satisfy $Z_M(\lambda_h) = 2\pi I_h$ for some chosen I_h (the set of I_h for $h = 1, \dots, H$ then labels the excited state we are examining). In the continuum limit this condition reads as $f(\theta_h) = 2i\pi I_h$. To numerically integrate the DdV equation we start from the vacuum evaluation of $f(\theta)$ and search for values in the θ complex plane where $f(\theta) = 2i\pi I_h$. We then insert this rough determination of θ_h 's in the DdV equation, and iterate to get a better determination of f . With this better determination we recompute the values of θ_h and then back to the DdV for a newer determination of f and so on iteratively. Although the convergence of this procedure slows down if compared to that of the vacuum case, it is still possible to determine the functions without much effort to some 7 or 8 significant digits.

5 The energy

The energy of the state as a functional of f can be obtained as the logarithm of the eigenvalue of the diagonal to diagonal transfer matrix corresponding to the state at exam

$$E_M = \frac{N}{R} \sum_{j=1}^M [\Phi(\lambda_j) - 2\pi] \quad (17)$$

where $\Phi(\lambda) = \phi(\lambda + \Theta, \frac{\gamma}{2}) - \phi(\lambda - \Theta, \frac{\gamma}{2})$. As before, we add and subtract a contribution from the holes

$$E_M = \frac{N}{R} \left[\sum_{j=1}^T \Phi(\lambda_j) - \sum_{h=1}^H \Phi(\lambda_h) - 2\pi M \right] \quad (18)$$

The first sum in this expression can be manipulated in a manner similar to what done in the deduction of DdV equation

$$\begin{aligned} \sum_{j=1}^T \Phi(\lambda_j) &= \oint_{\Gamma} \frac{d\mu}{2\pi i} \Phi(\mu) \frac{d}{d\mu} \log(1 + e^{iZ_M(\mu)}) \\ &= - \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \Phi'(x) Z_M(x) - i \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \Phi'(x + i\eta) \log(1 + e^{iZ_M(x+i\eta)}) \\ &\quad + i \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \Phi'(x - i\eta) \log(1 + e^{iZ_M(x-i\eta)}) \end{aligned} \quad (19)$$

Now we substitute the $Z_M(\mu)$ appearing in the first integral of this expression using the DdV equation (16). Also, rewrite the last term $2\pi M$ as $2\pi(N - \frac{H}{2})$. Introducing the notations

$$E_c = \frac{N^2}{R} \left(-2\pi + \int_{-\infty}^{+\infty} d\lambda \frac{\phi(\lambda + 2\Theta, \frac{\gamma}{2})}{\gamma \cosh \frac{\pi}{\gamma} \lambda} \right) \quad (20)$$

and

$$\psi(\lambda) = \frac{N}{\gamma R} [\text{sech}(\frac{\pi}{\gamma}(\lambda - \Theta)) - \text{sech}(\frac{\pi}{\gamma}(\lambda + \Theta))] \quad (21)$$

we can now write the energy as follows

$$\begin{aligned} E_M &= E_c - i \int_{-\infty}^{+\infty} d\lambda \psi(\lambda + i\eta) \log(1 + e^{iZ_M(\lambda+i\eta)}) \\ &\quad + i \int_{-\infty}^{+\infty} d\lambda \psi(\lambda - i\eta) \log(1 + e^{-iZ_M(\lambda-i\eta)}) \\ &\quad + i \frac{N}{R} \sum_{h=1}^H \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \Phi'(x) \log S(\frac{\pi}{\gamma}(x - \lambda_h)) + \frac{N}{R} H\pi - \frac{N}{R} \sum_{h=1}^H \Phi(\lambda_h) \end{aligned} \quad (22)$$

The first term E_c reproduces the standard bulk energy [4]. The second term, in the $N \rightarrow \infty$ limit acquires the same form as the vacuum energy in ref. [4]. Notice, however, that this similarity is only formal, because now this term is evaluated with the “excited” $f(\theta)$. The physical interpretation is that of an energy of the vacuum distorted by the presence of particle excitations. The last three terms are new and, when evaluated in the $N \rightarrow \infty$, $\Theta = \frac{\gamma}{2\pi} \log \frac{4N}{r} \rightarrow \infty$ limit, concur to give, after some algebra, a term of the kind $m \sum_{h=1}^H \cosh \theta_h$,

with the clear interpretation as a particle energy. Concluding, the continuum limit energy $\tilde{E}(R)$ of the excited state, epurated from the bulk energy and the infinities which are artifacts of the lattice, is

$$\tilde{E}(R) = \lim_{N \rightarrow \infty} (E_M - E_c) = m \sum_{h=1}^H \cosh \theta_h - \frac{m}{\pi} \int_{-\infty}^{+\infty} d\theta \sinh \theta \Im m \log(1 + e^{f(\theta+i0)}) \quad (23)$$

Note that in the infrared limit $r \rightarrow \infty$, where the convolution integral can be dropped, an H -hole state can be interpreted as an H -particle state. Having in our approach H even only, we see states with an even number of particles only. To deal with the more general problem of any number of particles, one should also consider DdV equations which are continuum limits over lattices with odd number of sites, or equivalently choose twisted boundary conditions in an appropriate way. This is, however, out of the scope of the present letter.

6 Kink DdV equations and analytic ultraviolet limit

The scaling function $\tilde{c}(r) = -6R\tilde{E}(R)/\pi$ is particularly interesting because, in the ultraviolet (UV) limit $r \rightarrow 0$, it provides the value of the conformal dimensions $\Delta_+ + \Delta_-$ and therefore allows the identification of the states with those of the CFT Hilbert space. The calculation of this limit can be done in an analytic way by resorting to the so called *kink* DdV equations. Let $x = \log(r/2)$. The UV limit is then given by $x \rightarrow -\infty$. The shape of the function $-if(\theta)$ in the region $x \ll 0$, is almost flat (and its value very close to α) inside the region $|\theta| \ll -x$ while it has a sharp $e^{r\theta}$ behaviour for $\theta \gg -x$ or $-e^{r\theta}$ for $\theta \ll x$. The hole positions θ_h are given by the condition $-if(\theta_h) = 2\pi I_h$. For $x \rightarrow -\infty$ therefore, $\theta_h \rightarrow \text{sign}(I_h)\infty$. Let us divide the set of holes into those with positive rapidity ($I_h > 0$) and those with negative rapidity ($I_h < 0$). Call these two sets \mathcal{I}_+ and \mathcal{I}_- respectively. We assume for simplicity that these two sets contain half of the holes each, $|\mathcal{I}_{\pm}| = H/2$. (Remember that if the number of sites of the lattice is even, the number of holes also must be even). The most general case will be treated in [6]. It is obvious that towards the UV limit, the holes in \mathcal{I}_+ (resp. \mathcal{I}_-) behave as right (resp. left) kinks. Their rapidities can be shifted as follows: $\rho_h = \theta_h \pm x$ if $h \in \mathcal{I}_{\pm}$. The DdV equation splits in the UV limit into two right (+) and left (-) *kink* DdV equations, correspondingly to the general shift $\theta \pm x = \rho$. Dropping terms exponentially small when $x \rightarrow -\infty$ and introducing the two kink pseudoenergies $f_{\pm}(\rho) \equiv \lim_{x \rightarrow -\infty} f(\pm(\rho - x))$, these two equations turn out to be scale invariant, i.e. independent of x

$$\begin{aligned} \mp if_{\pm}(\rho) &= e^{\pm\rho} \pm \alpha + \frac{H}{2}\alpha' \mp i \sum_{h \in \mathcal{I}_{\pm}} \log S(\rho - \rho_h) \\ &\pm 2 \int_{-\infty}^{+\infty} d\rho' G(\rho - \rho') \Im m \log(1 + e^{f_{\pm}(\rho+i0)}) \end{aligned} \quad (24)$$

where we have used the fact that $\log S(\pm\infty) = \pm i\alpha'$ with $\alpha' = \pi \frac{p-1}{2p}$. Correspondingly, the scaling function assumes the form $\tilde{c}(0) = c_+ + c_-$, with

$$c_{\pm} = -\frac{6}{\pi} \left[\sum_{h \in \mathcal{I}_{\pm}} e^{\pm \rho_h} \mp \frac{1}{\pi} \int_{-\infty}^{+\infty} d\rho e^{\pm \rho} \Im m \log(1 + e^{f_{\pm}(\rho+i0)}) \right] \quad (25)$$

Introducing $\varphi_{\pm}(\rho) \equiv e^{\pm \rho} \pm \alpha + \frac{H}{2}\alpha' \mp i \sum_{h \in \mathcal{I}_{\pm}} \log S(\rho - \rho_h)$ we can write eq.(25) as

$$\begin{aligned} c_{\pm} &= -\frac{6}{\pi} \sum_{h \in \mathcal{I}_{\pm}} e^{\pm \rho_h} + \frac{6}{\pi^2} \int_{-\infty}^{+\infty} d\rho \varphi'_{\pm}(\rho) \Im m \log(1 + e^{f_{\pm}(\rho+i0)}) \\ &\pm \frac{6}{\pi^2} \sum_{h \in \mathcal{I}_{\pm}} \int_{-\infty}^{+\infty} d\rho i \frac{d}{d\rho} \log S(\rho - \rho_h) \Im m \log(1 + e^{f_{\pm}(\rho+i0)}) \end{aligned} \quad (26)$$

The second term on the r.h.s. of this equation can be treated by resorting to the lemma cited in ref. [4] and observing that $f_{\pm}(\mp\infty + i0) = \pm i\alpha \frac{2p}{p+1}$. The first and third terms can be instead manipulated substituting $e^{\pm \rho_h}$ by use of the DdV equation and remembering that $\rho_h - \rho_{h'} \rightarrow 0$ for $x \rightarrow -\infty$ if both h, h' are in \mathcal{I}_+ or \mathcal{I}_- and $\log S(0) = 0$. One arrives at the final formula

$$\tilde{c}(0) = 1 - 12(\Delta_+ + \Delta_-) \quad (27)$$

with

$$\Delta_{\pm} = \frac{(\frac{H}{2}(p+1) \mp \kappa)^2}{4p(p+1)} + K_{\pm} \quad , \quad K_{\pm} = \pm \sum_{h \in \mathcal{I}_{\pm}} I_h - \frac{H^2}{8} \quad (28)$$

giving the left and right dimensions of the conformal state which is the UV limit of the H hole state we are examining. The (Lorentz) spin of the state is given by $\Delta_+ - \Delta_- = -\frac{H\kappa}{2p} + \sum_{h=1}^H I_h$. The numbers K_{\pm} are non-negative integers, and account for secondaries.

The states we are studying are uniquely defined by giving the sequence of (I_1, I_2, \dots, I_H) . Repetitions of values of I_h must be avoided in a sequence. As an example, we discuss here the case $p = 3$. The $\kappa = 0$ sector corresponds, at the UV limit, to conformal states selected by periodic boundary conditions. They are encoded in the modular invariant partition function of the model. An inspection of this latter shows that it contains a certain number of states of spin 0, among which there are the $\Delta_+ = \Delta_- = \frac{1}{3}, \frac{4}{3}, 3$ states. These can be obtained in our framework by exciting 2,4,6 holes respectively, with the lowest possible values of $|I_h|$

$$\left(-\frac{1}{2}, \frac{1}{2}\right) \rightarrow \frac{1}{3} \quad , \quad \left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right) \rightarrow \frac{4}{3} \quad , \quad \left(-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right) \rightarrow 3 \quad (29)$$

Using larger values of $|I_h|$, one can get secondaries of these states, even with spin different from zero. Often, there are many degenerate secondaries with the same values of $\Delta_+ + \Delta_-$. We can also obtain the same values of $\Delta_+ + \Delta_-$ by different sequences of I_h 's. In the example above 2-hole states and 4-hole states often degenerate in the conformal limit, although their infrared destiny is very different, going to 2-particle and 4-particle states respectively. To link the description of this degenerations in terms of hole states with the Virasoro (or extended conformal algebra) description is a very interesting problem to investigate.

Of course the modular invariant partition function mentioned before, or its analogs for other values of p , contain other states of spin 0 that we cannot reach with our construction as far as we do not deal with string excitations (see [6]). The hope is, once both hole and string states are taken into account, to show that the whole modular invariant partition function can be reconstructed.

If one considers now hole excitations over the other α -vacua with $\kappa \neq 0$, one should expect to reconstruct, in the UV limit, the partition functions with α -twisted boundary conditions. It is a straightforward exercise of CFT to write down these partition functions and compare the operator content of these sectors with our results. The states we find by choosing 2,4,6... holes with the “minimal” choice of I_h ’s have now fractional spins. It is a matter of fact that such exotic states are indeed present in the twisted partition functions.

7 Numerical work

The results of the previous section have been checked by numerical integration of the DdV equations. Apart the already mentioned numerical study of the α -vacua, that confirms with high precision the expected identification with RSG_p excited states on the diagonal of the Kac-table – a new result by itself – we have investigated numerically the hole excitations with 2,4,6 holes for values of $p = 3, 4, 5$. The precision, in this investigation has been taken a bit lower, say 6 or 7 significant digits. It is a simple choice of speed, no technical problem prevents to get higher precision data. A detailed summary of these numerical results, together with their particle interpretation, level crossing phenomena, etc... will be given in [6]. Here we only present, as an example of the method, the energy evolution against R of few of the two and four holes spin zero excited states over the $\alpha = 0$ vacuum in $p = 3$. The self explanatory Fig.1 collects these data. One can clearly see that the H -hole states accumulate to H -particle states in the $R \rightarrow \infty$ limit, as commented before.

It is interesting to note the undoubted existence, in this graph, of a level crossing between the $(-\frac{5}{2}, \frac{5}{2})$ and the $(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ states at $R = 5.613...$. Such level crossings must be present in an integrable model. One of the advantages of knowing scaling functions exactly, compared to other approximate methods of investigation like e.g. TCSA, is that the existence of a level crossing can be established with no doubt. There is no danger here to confuse a level crossing with two lines that approach very close and then repulse each other without crossing: each line, in the DdV approach, is computed independently from the others.

Of course there are many states missing in Fig.1. We have not investigated all the states that include strings in their Bethe root configurations. These should complete the picture and most probably provide for more level crossings. We have also done some investigation in the other α -sectors, that we do not reproduce here for pure reasons of space. In these sectors the number of spin 0 states is much lower. It is interesting to notice the existence of fractional spin states. The $\alpha = \pi$ sector is particularly interesting, because it realizes the antiperiodic boundary conditions where it should be possible to see the spin $\frac{1}{2}$ states corresponding to the

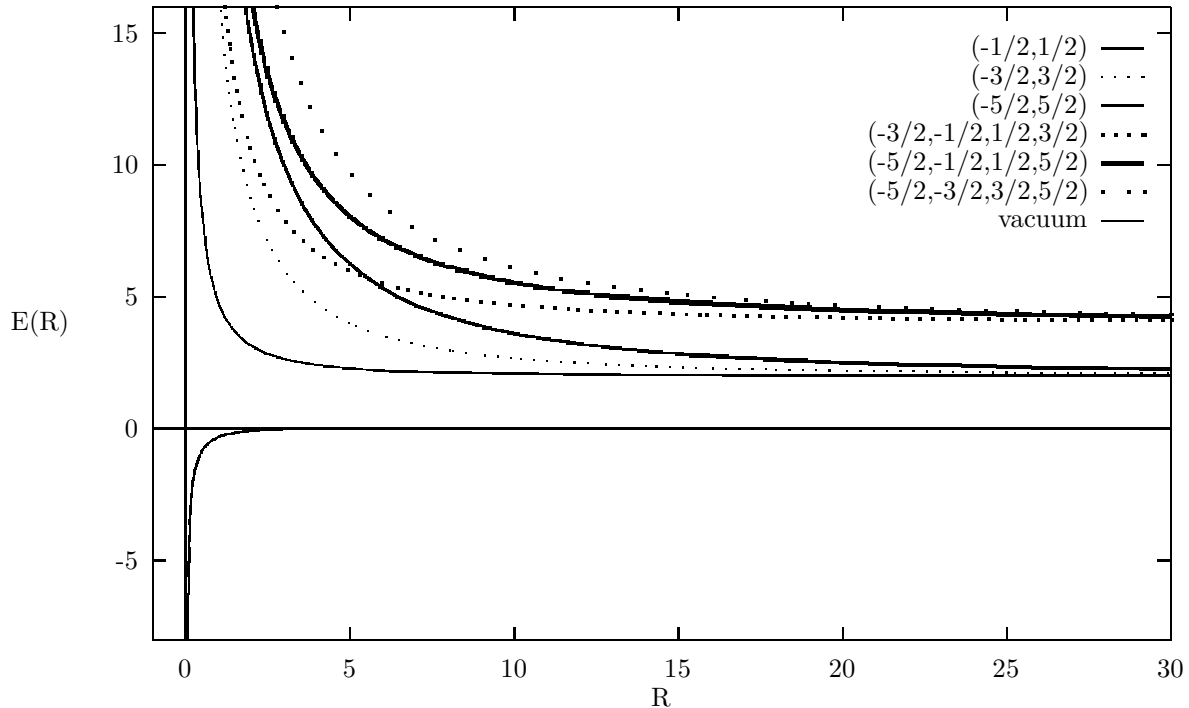


Figure 1: Energy versus R of spin 0 pure hole excitations in the $\alpha = 0$ sector of $\text{SG}_{p=3}$ model.

Thirring fermions.

8 Conclusions

We have considered here hole excitations of the SG model over the α -vacua, finding a DdV equation for them and calculating their exact scaling functions, numerically for generic R and analitically for $R \rightarrow 0$. In this latter limit, we have found agreement with the operator content dictated by the CFT partition functions in the variuos twisted sectors. Of course, we have reproduced only a part of the spectrum, because we have not considered more complicated states including strings. We intend to deal with the general problem in a forthcoming publication [6]. In this short letter we made the choice to deal with this technically simpler problem to probe the method and show its efficiency. The only states we have provided for RSG_p models through q-reduction are for the moment those corresponding to the alpha vacua, i.e. the off-critical deformations of the *spin* primary states which lie on the diagonal of the Kac-table. It is of course obvious that once the full set of SG states will be treated, access to the scaling functions for all the RSG_p states will also be available. The extension of the method to non-integer rational p could then allow the treatment of non-unitary minimal models perturbed by ϕ_{13} too. Among these, the Lee-Yang model plays the particular role of being the simplest model, on which both TCSA data and recent TBA exploration of few excited states are available [17, 18]. It would be extremely interesting to make a comparison

between these results and our approach. In particular the method of ref. [18] suggests to explore monodromy issues of the scaling functions also in our approach. The link between monodromies and Bethe ansatz interpretation of the results would certainly provide more insight in the problem of excited states in general.

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